

Convertibility of Observables

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Abstract

Some problems of quantum information, cloning, estimation and testing of states, universal coding etc., are special example of the following ‘state convertibility’ problem. In this paper, we consider the dual of this problem, ‘observable conversion problem’. Given families of operators $\{L_\theta\}_{\theta \in \Theta}$ and $\{M_\theta\}_{\theta \in \Theta}$, we ask whether there is a completely positive (sub) unital map which sends L_θ to M_θ for each θ . We give necessary and sufficient conditions for the convertibility in some special cases.

1 Introduction

1.1 Problem treated in the paper

Some problems of quantum information, cloning, estimation and testing of states, universal coding etc., are special example of the following ‘state convertibility’ problem. Consider parameterized families of density operators $\mathcal{E} = \{\rho_\theta\}_{\theta \in \Theta}$, $\mathcal{F} = \{\sigma_\theta\}_{\theta \in \Theta}$, each on a finite dimensional Hilbert space \mathcal{H} and \mathcal{K} , respectively. Then the question is whether there is a completely trace preserving positive (CPTP) map Φ such that

$$\forall \theta \in \Theta, \|\Phi(\rho_\theta) - \sigma_\theta\|_1 \leq e_\theta, \quad (1)$$

where e_θ are small non-negative numbers. Its errorless version is to find whether there is a CPTP map Λ such that

$$\forall \theta \in \Theta, \Phi(\rho_\theta) = \sigma_\theta. \quad (2)$$

In this paper we consider its ‘dual’ problem, or ‘observable convertibility’ problem. Denote by $\mathcal{L}(\mathcal{H})$ the set of all the linear operators over \mathcal{H} ($\dim \mathcal{H} < \infty$ unless otherwise mentioned) and $I_{\mathcal{H}}$ is the identity element of $\mathcal{L}(\mathcal{H})$. Consider sets of positive operators,

$$\hat{\mathcal{E}} = \{L_\theta\}_{\theta \in \Theta}, \hat{\mathcal{F}} = \{M_\theta\}_{\theta \in \Theta},$$

where $|\Theta| < \infty$ ($\Theta = \{1, \dots, |\Theta|\}$) and L_θ and M_θ is acting on d -dimensional space \mathcal{H} and d' -dimensional space \mathcal{K} . respectively. Our question is whether there is a complete positive (CP) (sub)unital or unital map Λ such that

$$\Lambda(L_\theta) = M_\theta, \forall \theta \in \Theta. \quad (3)$$

1.2 Motivation

One application of the problem treated here is the question of the order structure of POVMs treated in [3]: If POVM $\{E_i\}_{i \in I}$ and $\{F_i\}_{i \in I}$ satisfies

$$\forall i \in I, \Lambda(E_i) = F_i$$

for certain CP unital map Λ , the latter can be made from the former by a physical transformation. Thus $\{E_i\}_{i \in I}$ is more useful than $\{F_i\}_{i \in I}$ for any tasks, obviously. To check the relation holds, one instead can check (3) by setting $\Theta = I \setminus \{i_0\}$, and $L_\theta = c_\theta E_\theta$, $M_\theta = c_\theta F_\theta$, where $c_\theta \in \mathbb{R}$. In this case, Λ considered is a CP unital map Λ .

Sometimes, we are interested in sub-POVMs, or sets of positive operators with

$$\sum_{\theta \in \Theta} L_\theta \leq I_{\mathcal{H}}.$$

For example, in case of detection of unknown states $\{\rho_\theta\}_{\theta \in \Theta}$, sometimes we allow the answer "I don't know". Then, if L_θ corresponds to the answer ' ρ_θ is the true state', the sum $\sum_{\theta \in \Theta} L_\theta$ is smaller than or equal to $I_{\mathcal{H}}$, and $I_{\mathcal{H}} - \sum_{\theta \in \Theta} L_\theta$ corresponds to "I don't know". In this case, transformation by a CP subunital map is of interest. Suppose (3) holds for a CP subunital map Λ and $\sum_{\theta \in \Theta} M_\theta$ is smaller than or equal to $I_{\mathcal{K}}$. Then the measurement corresponding to $\{M_\theta\}_{\theta \in \Theta}$ is realized by the one corresponding to $\{L_\theta\}_{\theta \in \Theta}$ in the following manner. Given an input state ρ , we perform the measurement

$$\rho \rightarrow \begin{cases} \sqrt{I_{\mathcal{K}} - \Lambda^*(I_{\mathcal{H}})} \rho \sqrt{I_{\mathcal{K}} - \Lambda^*(I_{\mathcal{H}})}, & \text{output = 'I don't know'}, \\ \Lambda(\rho), & \text{output = 'proceed'}. \end{cases}$$

If the measurement result is 'proceed' we apply the measurement corresponding to $\{L_\theta\}_{\theta \in \Theta}$.

Also, (3) is related to the 'state conversion' problem. Let us define $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$ by

$$\sum_{\theta \in \Theta} \rho_\theta = S S^\dagger, \quad \sum_{\theta \in \Theta} \sigma_\theta = T T^\dagger, \quad (4)$$

and define

$$L_\theta := S^{-1} \rho_\theta S^{\dagger-1}, \quad M_\theta := T^{-1} \sigma_\theta T^{\dagger-1}. \quad (5)$$

(Note that in 'state conversion' problem, we can suppose $\text{supp } S = \mathcal{H}$ and $\text{supp } T = \mathcal{K}$ without loss of generality.) If (2) holds for a CP trace preserving map Φ , the map

$$\Lambda(X) := T^{-1} \Phi(S X S^\dagger) T^{\dagger-1} \quad (6)$$

is CP and unital, and Λ satisfies (3) and

$$\Lambda^* (T^\dagger T) = S^\dagger S. \quad (7)$$

So existence of a CP unital map with (3) is necessary condition for existence of a CPTP map with (2).

Conversely, if (3) holds for a CP unital map Λ , the CP map Φ defined by

$$\Phi(X) := T \Lambda(S^{-1} X S^{\dagger-1}) T^\dagger \quad (8)$$

satisfies (2). Φ is trace preserving if and only if (7) holds.

Another link to state convertibility problem is as follows. It is known that the existence of CPTP map with (2) is equivalent to, when $|\Theta| < \infty$,

$$\inf_{\Lambda} \sum_{\theta \in \Theta} \text{tr} \Lambda(L_\theta) \rho_\theta p_\theta \leq \sum_{\theta \in \Theta} \text{tr} L_\theta \sigma_\theta p_\theta$$

holds for any parameterized family of positive operators $\{L_\theta\}_{\theta \in \Theta}$ with $\|L_\theta\| \leq 1$ and for any probability distributions p_θ on Θ . Here, Λ moves all over the CP trace preserving maps, or all over the CP trace non-increasing maps. To solve this problem, the knowledge about $\{\Lambda^*(L_\theta)\}_{\theta \in \Theta}$ when Λ^* moves over all the CP (sub)unital maps will be of some help.

1.3 Notations, conventions, and a small technical point

Here we add some more notations used in the paper. In this paper $d = \dim \mathcal{H} < \infty$ unless otherwise mentioned. A map Λ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ is said to be unital if $\Lambda(I_{\mathcal{H}}) = I_{\mathcal{K}}$, and subunital if $\Lambda(I_{\mathcal{H}}) \leq I_{\mathcal{K}}$. By definition, any unital map is subunital. $P_{\mathcal{H}}$ is the projection onto the vector space \mathcal{H} . $\|A\|$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ denotes the operator norm, the largest eigenvalue, and the smallest eigenvalue, respectively. Also, $\text{sp}(A) := \lambda_{\max}(A) - \lambda_{\min}(A)$. $\|A\|_1$ is the trace norm of A , $\|A\|_1 := \text{tr} \sqrt{A^\dagger A}$. For a matrices $A = [A_{i,j}]$ and $B = [B_{i,j}]$, the Hadamard product $A \circ B$ is defined by $(A \circ B)_{i,j} = A_{i,j} B_{i,j}$.

In dealing with ‘observable convertibility’, there is a subtle point which was absent in ‘state convertibility’ problem. In the latter, the input Hilbert space \mathcal{H} and the output Hilbert space \mathcal{K} could be any space which contains $\sum_{\theta \in \Theta} \text{supp} \rho_\theta$ and $\sum_{\theta \in \Theta} \text{supp} \sigma_\theta$, respectively. This is not the case in case that Λ is a unital map. The reason is as follows.

Let Λ be a linear map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, and let \mathcal{H}' and \mathcal{K}' be Hilbert spaces with $\mathcal{H}' \subset \mathcal{H}$ and $\mathcal{K} \subset \mathcal{K}'$. Then the restriction of Λ to $\mathcal{L}(\mathcal{H}')$ nor the imbedding the range of Λ into \mathcal{K}' is not unital in general. So even if there is a CP map with (3) and $\Lambda(I_{\mathcal{H}}) = I_{\mathcal{K}}$, there might not be any Λ with and $\Lambda(I_{\mathcal{H}'}) = I_{\mathcal{K}'}$. Therefore, not only the sets of the observables $\hat{\mathcal{E}} = \{L_\theta\}_{\theta \in \Theta}$ and $\hat{\mathcal{F}} = \{M_\theta\}_{\theta \in \Theta}$, the choice of underlying Hilbert spaces \mathcal{H} and \mathcal{K} is important part of the problem.

In dealing with problem, an easy and useful necessary condition for (3) is $\Lambda(L_\theta) = M_\theta$ for each $\theta \in \Theta_0$, where Θ_0 is a subset of Θ . In case that

$\sum_{\theta \in \Theta_0} \text{supp } L_\theta$ is strictly smaller than $\sum_{\theta \in \Theta} \text{supp } L_\theta$, one may be tempted to replace \mathcal{H} by $\sum_{\theta \in \Theta_0} \text{supp } L_\theta$. But this is not possible, as mentioned above. This is one reason why we also pay attention to conversion by subunital map. In this case one can freely chose underlying Hilbert space, giving tractable necessary conditions for existence of a unital map with (3).

2 An application to a ‘state conversion’ problem

Suppose

$$\rho_\theta = |u_\theta\rangle \langle u_\theta|, \quad \sigma_\theta = |v_\theta\rangle \langle v_\theta|,$$

(In the what follows, we do not assume $\text{tr } \rho_\theta = \text{tr } \sigma_\theta = 1$. Thus u_θ and v_θ may not be normalized.) Denote by \mathcal{U} and \mathcal{V} the family $\{u_\theta\}_{\theta \in \Theta}$ and $\{v_\theta\}_{\theta \in \Theta}$, respectively. Denote by $G_{\mathcal{U}}$ and $G_{\mathcal{V}}$ the Gram matrix of \mathcal{U} and \mathcal{V} , respectively, that is,

$$G_{\mathcal{U}, \theta, \theta'} := \langle u_\theta | u_{\theta'} \rangle, \quad G_{\mathcal{V}, \theta, \theta'} := \langle v_\theta | v_{\theta'} \rangle.$$

Theorem 1 (Theorem 2 of [4]) *There is a CP trace preserving map Φ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ satisfying (2) if and only if there is a matrix $H = [H_{\theta, \theta'}]$ such that*

$$G_{\mathcal{U}, \theta, \theta'} = G_{\mathcal{V}, \theta, \theta'} \circ H, \quad (9)$$

$$H \geq 0, \quad H_{\theta, \theta} = 1. \quad (10)$$

Theorem 2 (Corollary 1 of [4]) *There is a CP trace preserving map Φ from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ satisfying (3) and Φ' from $\mathcal{L}(\mathcal{K})$ to $\mathcal{L}(\mathcal{H})$ satisfying*

$$\Phi'(\sigma_\theta) = \rho_\theta, \forall \theta \in \Theta \quad (11)$$

if and only if \mathcal{U} and \mathcal{V} are unitary equivalent.

Remark 3 *In Theorem 2 of [4], they do not have condition that $H_{\theta, \theta} = 1$. In their case, they consider mapping from a family of normalized vectors to another family of normalized vectors, so that $H_{\theta, \theta} = 1$ holds automatically. In our case, we have to impose this additional constrain because the system of vectors may not be normalized.*

To show link between ‘state conversion’ and ‘state conversion’, we give another proof of Theorems 1-2, in case that \mathcal{U} and \mathcal{V} are linearly independent.

Suppose $\text{supp } \sum_{\theta \in \Theta} \rho_\theta = \mathcal{H}$ and $\text{supp } \sum_{\theta \in \Theta} \sigma_\theta = \mathcal{K}$ without loss of generality, so that $\dim \mathcal{H} = \dim \mathcal{K} = |\Theta|$. Define $S \in \mathcal{L}(\mathcal{H})$ and $T \in \mathcal{L}(\mathcal{K})$ by

$$S = \sum_{\theta \in \Theta} |u_\theta\rangle \langle e_\theta|, \quad T = \sum_{\theta \in \Theta} |v_\theta\rangle \langle f_\theta|,$$

where $\{e_\theta\}_{\theta \in \Theta}$ and $\{f_\theta\}_{\theta \in \Theta}$ is a complete orthonormal basis of \mathcal{H} and \mathcal{K} , respectively. Then S and T satisfy (4). It is easy to check $\{L_\theta\}_{\theta \in \Theta}$ and $\{M_\theta\}_{\theta \in \Theta}$ defined by (5) are orthonormal projections,

$$L_\theta = |e_\theta\rangle \langle e_\theta|, \quad M_\theta = |f_\theta\rangle \langle f_\theta|.$$

Suppose there is a CPTP map with (2). Then, a CP map Λ defined by (6) is unital and satisfies (3), so it is a ‘dephasing map’,

$$\Lambda(|e_\theta\rangle \langle e_{\theta'}|) = H_{\theta, \theta'} |f_\theta\rangle \langle f_{\theta'}|, \quad (12)$$

where $H = [H_{\theta, \theta'}]$ satisfies (10). It is easy to see

$$\Lambda^*(|f_\theta\rangle \langle f_{\theta'}|) = H_{\theta, \theta'} |e_\theta\rangle \langle e_{\theta'}|.$$

Also Λ satisfies (7),

$$\begin{aligned} \Lambda^* \left(\sum_{\theta, \theta' \in \Theta} G_{\mathcal{V}, \theta, \theta'} |f_\theta\rangle \langle f_{\theta'}| \right) &= \sum_{\theta, \theta' \in \Theta} H_{\theta, \theta'} G_{\mathcal{V}, \theta, \theta'} |e_\theta\rangle \langle e_{\theta'}| \\ &= \sum_{\theta, \theta' \in \Theta} G_{\mathcal{U}, \theta, \theta'} |e_\theta\rangle \langle e_{\theta'}|. \end{aligned}$$

Therefore, (9) is necessary.

Conversely, if (9) holds, the map defined by (12) is a CP unital map, and satisfies (3) and (7). Also, the CP map defined by (8) is trace-preserving and satisfies (2). Thus (9) is sufficient. Thus we obtain Theorems 1 in case that \mathcal{U} and \mathcal{V} are linearly independent.

Next, suppose there is a CPTP map Φ' with (11) exists. We also suppose $|e_\theta\rangle = |f_\theta\rangle$ without loss of generality. Then, by [9], the map

$$\Phi'(X) := U_2 \Lambda^* (U_1^\dagger X U_1) U_2^\dagger$$

should be an example of such a map, where U_1 and U_2 are unitary operators defined by

$$\sqrt{TT^\dagger} U_1 = T, \quad \sqrt{SS^\dagger} U_2 = S.$$

Then Φ' maps a pure state to another pure state only if Λ^* does so. In turn, Λ^* maps a pure state to another pure state only if it does not change the pure state. Therefore, we have Theorems 2 in case that \mathcal{U} and \mathcal{V} are linearly independent.

3 Conversion between rank-1 operators

In this section,

$$L_\theta = |u_\theta\rangle \langle u_\theta|, \quad M_\theta = |v_\theta\rangle \langle v_\theta|.$$

Let $\mathcal{U}^\dagger := \{u_\theta^\dagger\}_{\theta \in \Theta}$ and $\mathcal{V}^\dagger := \{v_\theta^\dagger\}_{\theta \in \Theta}$ be the dual system (if exists) of \mathcal{U} and \mathcal{V} , respectively,

$$\langle u_\theta^\dagger | u_{\theta'} \rangle = \delta_{\theta, \theta'}, \quad \langle v_\theta^\dagger | v_{\theta'} \rangle = \delta_{\theta, \theta'}.$$

Lemma 4 Suppose that \mathcal{V} is linearly independent. Then a positive operator C supported on $\text{span}\mathcal{V}$ is identical to $|v_\theta\rangle\langle v_\theta|$ if and only if

$$\langle v_{\theta'}^\uparrow | C | v_{\theta'}^\uparrow \rangle = \delta_{\theta,\theta'}. \quad (13)$$

Also, there is a CP unital map Λ satisfying (3). Then \mathcal{U} is also linearly independent.

Proof. If C is identical to $|v_\theta\rangle\langle v_\theta|$, (13) is trivially true. If (13) holds, the positive operator C has to have null space spanned by $\{v_{\theta'}^\uparrow; \theta' \in \Theta, \theta' \neq \theta\}$. Therefore, C is constant multiple of $|v_\theta\rangle\langle v_\theta|$. The constant factor is fixed by the condition $\langle v_{\theta'}^\uparrow | C | v_{\theta'}^\uparrow \rangle = 1$. Thus, we have the first assertion.

(3) holds only if

$$\begin{aligned} \text{tr } \Lambda^* \left(|v_{\theta_0}^\uparrow\rangle\langle v_{\theta_0}^\uparrow| \right) |u_\theta\rangle\langle u_\theta| &= \text{tr } |v_{\theta_0}^\uparrow\rangle\langle v_{\theta_0}^\uparrow| \Lambda(|u_\theta\rangle\langle u_\theta|) = 0, \theta \neq \theta_0, \\ \text{tr } \Lambda^* \left(|v_{\theta_0}^\uparrow\rangle\langle v_{\theta_0}^\uparrow| \right) |u_{\theta_0}\rangle\langle u_{\theta_0}| &= \text{tr } |v_{\theta_0}^\uparrow\rangle\langle v_{\theta_0}^\uparrow| \Lambda(|u_{\theta_0}\rangle\langle u_{\theta_0}|) = 1, \end{aligned}$$

These lead to contradiction if u_{θ_0} is in the span of \mathcal{U} . Therefore, \mathcal{U} should be linearly independent. ■

The proof of the following lemma is almost immediate.

Lemma 5 There is a CP map Λ satisfying (3) holds if and only if

$$\exists \alpha_{\theta,i}, \quad \alpha_{\theta,i} |v_\theta\rangle = W_i |u_\theta\rangle, \quad \sum_i |\alpha_{\theta,i}|^2 = 1, \quad (14)$$

where W_i 's are Kraus operators of Λ .

The following theorem is almost a dual of Theorem 1.

Theorem 6 Suppose that \mathcal{V} is linearly independent. Then there is a CP sub-unital map Λ^* satisfying (3) if and only if there is a matrix $H = [H_{\theta,\theta'}]$ such that

$$G_{\mathcal{V}}^{-1} \geq H \circ G_{\mathcal{U}}^{-1}, \quad (15)$$

$$H \geq 0, H_{\theta,\theta} = 1, \theta \in \Theta, \quad (16)$$

Proof. By Lemma 4, $\mathcal{U} = \{|u_\theta\rangle\}_{\theta \in \Theta}$ is also linearly independent. Without loss of generality, we suppose $\mathcal{H} = \text{span}\mathcal{U}$ and $\mathcal{K} = \text{span}\mathcal{V}$, and thus $d = d' = |\Theta|$. Let $\{W_i\}$ be Kraus operators of Λ , $\Lambda(L) = \sum_i W_i L W_i^\dagger$.

Let us denote by $[\mathcal{U}]$ and $[\mathcal{V}]$ the matrix whose θ 's column vector is $|u_\theta\rangle$ and $|v_\theta\rangle$, respectively. Observe they are square matrices and invertible. So the condition (14) is rewritten as

$$[\mathcal{V}] \text{diag}(\alpha_{1,i}, \dots, \alpha_{|\Theta|,i}) = W_i [\mathcal{U}],$$

or equivalently

$$W_i = [\mathcal{V}] \text{diag} (\alpha_{1,i}, \dots, \alpha_{|\Theta|,i}) [\mathcal{U}]^{-1}. \quad (17)$$

A CP map Λ^* given by the Kraus operators (17) is subunital if and only if

$$\begin{aligned} I_{\mathcal{K}} &\geq \sum_i W_i W_i^\dagger = \sum_i [\mathcal{V}] \text{diag} (\alpha_{1,i}, \dots, \alpha_{|\Theta|,i}) [\mathcal{U}]^{-1} [\mathcal{U}]^{-1\dagger} \text{diag} (\overline{\alpha_{1,i}}, \dots, \overline{\alpha_{|\Theta|,i}}) [\mathcal{V}]^\dagger \\ &= \sum_i [\mathcal{V}] \text{diag} (\alpha_{1,i}, \dots, \alpha_{|\Theta|,i}) G_{\mathcal{U}}^{-1} \text{diag} (\overline{\alpha_{1,i}}, \dots, \overline{\alpha_{|\Theta|,i}}) [\mathcal{V}]^\dagger \\ &= [\mathcal{V}] H \circ G_{\mathcal{U}}^{-1} [\mathcal{V}]^\dagger, \end{aligned}$$

where $H_{\theta, \theta'} := \sum_i \alpha_{\theta,i} \overline{\alpha_{\theta',i}}$. Since H satisfies (16) and $[\mathcal{V}]^{-1} [\mathcal{V}]^{-1\dagger} = G_{\mathcal{V}}^{-1}$, we have the assertion. ■

The following theorem can be proved in almost parallel manner as the previous theorem. But to make the relation with Theorem 1, we give the proof using Theorem 1.

Theorem 7 *Suppose that \mathcal{V} is linearly independent, and $\text{span } \mathcal{V} = \mathcal{K}$, $\text{span } \mathcal{U} = \mathcal{H}$. Then there is a CP unital map Λ^* satisfying (3) if and only if there is a matrix $H = [H_{\theta, \theta'}]$ with (16) and*

$$G_{\mathcal{V}}^{-1} = H \circ G_{\mathcal{U}}^{-1}. \quad (18)$$

Proof. Since by Lemma 4, $\{|u_\theta\rangle\}_{\theta \in \Theta}$ is also linearly independent, $|\Theta| = d = d'$. By Lemma 4, (3) is same as

$$\delta_{\theta, \theta'} = \text{tr} \left| v_{\theta'}^\uparrow \right\rangle \left\langle v_{\theta'}^\uparrow \right| \Lambda (|u_\theta\rangle \langle u_\theta|) = \text{tr} \Lambda^* \left(\left| v_{\theta'}^\uparrow \right\rangle \left\langle v_{\theta'}^\uparrow \right| \right) |u_\theta\rangle \langle u_\theta|.$$

Therefore, replacing v_θ and v_θ^\uparrow in (13) by u_θ and u_θ^\uparrow respectively, we have

$$\Lambda^* \left(\left| v_{\theta'}^\uparrow \right\rangle \left\langle v_{\theta'}^\uparrow \right| \right) = \left| u_{\theta'}^\uparrow \right\rangle \left\langle u_{\theta'}^\uparrow \right|.$$

Thus, we can use Theorem 1. Noticing $G_{\mathcal{U}\uparrow} = G_{\mathcal{U}}^{-1}$ and $G_{\mathcal{V}\uparrow} = G_{\mathcal{V}}^{-1}$, we obtain the asserted condition. ■

Below, $[G_{\mathcal{U}}]_{\Theta_1, \Theta_2}$ means submatrix of $G_{\mathcal{U}}$ that corresponds to the rows with index in Θ_1 and the columns with index in Θ_2 . Also, \mathcal{U}_{Θ_1} , \mathcal{V}_{Θ_1} , $\hat{\mathcal{E}}_{\Theta_1}$, and $\hat{\mathcal{F}}_{\Theta_1}$ means restriction of the range of θ to Θ_1 of \mathcal{U} , \mathcal{V} , $\hat{\mathcal{E}}$, and $\hat{\mathcal{F}}$.

Lemma 8 *Suppose that \mathcal{U} and \mathcal{V} satisfy all the conditions of Theorem 6. Then, there is a CP unital map Λ satisfying (3) only if*

$$\det [G_{\mathcal{U}}]_{\Theta_1, \Theta_1} \geq \det [G_{\mathcal{V}}]_{\Theta_1, \Theta_1} \quad (19)$$

for any $\Theta_1 \subset \Theta$.

Proof. We show the assertion using induction about $|\Theta|$. When $|\Theta| = 1$,

$$\det [G_{\mathcal{U}}] = \| |u_1\rangle \langle u_1| \|_\infty.$$

Thus by (44), we have the assertion.

Next, suppose that the assertion is true for Θ_1 such that $|\Theta_1| \leq |\Theta| - 1$. (Without loss of generality, $\Theta_1 \subset \Theta$.) Observe that the any subfamily \mathcal{U}_{Θ_1} and \mathcal{V}_{Θ_1} satisfy all the conditions of Theorem 6. (Here we replace Θ in (3) with Θ_1 .) Then by the hypothesis of the induction, (19) holds for any subset $\Theta_1 \neq \Theta$. So it remains to show (19) for $\Theta_1 = \Theta$. Since

$$(G_{\mathcal{U}}^{-1})_{\theta, \theta} = \frac{\det [G_{\mathcal{U}}]_{\Theta_2, \Theta_2}}{\det G_{\mathcal{U}}},$$

where $\Theta_2 = \Theta \setminus \{\theta\}$, Theorem 6 implies that

$$\frac{\det [G_{\mathcal{V}}]_{\Theta_2, \Theta_2}}{\det G_{\mathcal{V}}} \geq \frac{\det [G_{\mathcal{U}}]_{\Theta_2, \Theta_2}}{\det G_{\mathcal{U}}}.$$

Since $\det [G_{\mathcal{V}}]_{\Theta_2, \Theta_2} \leq \det [G_{\mathcal{U}}]_{\Theta_2, \Theta_2}$ by the hypothesis of the induction, we have $\det G_{\mathcal{U}} \geq \det G_{\mathcal{V}}$. Thus we have the assertion. ■

Theorem 9 *Suppose that \mathcal{U} and \mathcal{V} satisfy all the conditions of Theorem 6 and $|\Theta| = 2$. Then, there is a CP unital map Λ^* satisfying (3) if and only if*

$$\frac{\|v_2\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_2\|^2}{\det G_{\mathcal{U}}} \geq 0, \frac{\|v_1\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_1\|^2}{\det G_{\mathcal{U}}} \geq 0, \quad (20)$$

$$\left(\frac{|\langle v_1 | v_2 \rangle|}{\det G_{\mathcal{V}}} - \frac{|\langle u_1 | u_2 \rangle|}{\det G_{\mathcal{U}}} \right)^2 \leq \left(\frac{\|v_2\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_2\|^2}{\det G_{\mathcal{U}}} \right) \left(\frac{\|v_1\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_1\|^2}{\det G_{\mathcal{U}}} \right). \quad (21)$$

Proof. By Theorem 6, existence of a CP unital map Λ with (3) is equivalent to the existence of a complex number η with $|\eta| \leq 1$ and

$$\left[\begin{array}{c} \frac{\|v_2\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_2\|^2}{\det G_{\mathcal{U}}} - \left(\frac{\langle v_1 | v_2 \rangle}{\det G_{\mathcal{V}}} - \eta \frac{\langle u_1 | u_2 \rangle}{\det G_{\mathcal{U}}} \right) \\ - \left(\frac{\langle v_1 | v_2 \rangle}{\det G_{\mathcal{V}}} - \eta \frac{\langle u_1 | u_2 \rangle}{\det G_{\mathcal{U}}} \right) \frac{\|v_1\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_1\|^2}{\det G_{\mathcal{U}}} \end{array} \right] \geq 0,$$

or equivalently, (20) and

$$\left| \frac{\langle v_1 | v_2 \rangle}{\det G_{\mathcal{V}}} - \eta \frac{\langle u_1 | u_2 \rangle}{\det G_{\mathcal{U}}} \right|^2 \leq \left(\frac{\|v_2\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_2\|^2}{\det G_{\mathcal{U}}} \right) \left(\frac{\|v_1\|^2}{\det G_{\mathcal{V}}} - \frac{\|u_1\|^2}{\det G_{\mathcal{U}}} \right).$$

Maximizing the LHS of the this expression moving η so that $|\eta| \leq 1$, we obtain (21). ■

Lemma 10 *Suppose that $\|u_{\theta}\| = \|v_{\theta}\|$ for each $\theta \in \Theta$. Then there is a CP subunital map Λ satisfying (3) exists only if*

$$L_{\theta} = |u_{\theta}\rangle \langle u_{\theta}| \in \mathcal{M}_{\Lambda}$$

for each $\theta \in \Theta$.

Proof. Since Λ is subunital, its norm does not exceeds 1 by Lemma 22. Since

$$\frac{\|\Lambda(|u_\theta\rangle\langle u_\theta|)\|}{\||u_\theta\rangle\langle u_\theta|\|} = \frac{\||v_\theta\rangle\langle v_\theta|\|}{\||u_\theta\rangle\langle u_\theta|\|},$$

$\|\Lambda\| = 1$. Hence, since

$$\begin{aligned}\Lambda(|u_\theta\rangle\langle u_\theta| \cdot |u_\theta\rangle\langle u_\theta|) &= \|u_\theta\|^2 \Lambda(|u_\theta\rangle\langle u_\theta|) \\ &= \|u_\theta\|^2 |v_\theta\rangle\langle v_\theta| = \|v_\theta\|^2 |v_\theta\rangle\langle v_\theta| \\ &= |v_\theta\rangle\langle v_\theta| \cdot |v_\theta\rangle\langle v_\theta|,\end{aligned}$$

$|u_\theta\rangle\langle u_\theta|$ is an element of \mathcal{M}_Λ for each $\theta \in \Theta$. ■

Theorem 11 *Suppose that $\|u_\theta\| = \|v_\theta\|$ for each $\theta \in \Theta$. Then there is a CP subunital map Λ satisfying (3) exists if and only if they are unitary equivalent.*

Proof. By Lemma 10,

$$\begin{aligned}\langle u_\theta | u_{\theta'} \rangle \Lambda(|u_\theta\rangle\langle u_{\theta'}|) &= \Lambda(|u_\theta\rangle\langle u_\theta| \cdot |u_{\theta'}\rangle\langle u_{\theta'}|) = \Lambda(|u_\theta\rangle\langle u_\theta|) \Lambda(|u_{\theta'}\rangle\langle u_{\theta'}|) \\ &= |v_\theta\rangle\langle v_\theta| \cdot |v_{\theta'}\rangle\langle v_{\theta'}| = \langle v_\theta | v_{\theta'} \rangle |v_\theta\rangle\langle v_{\theta'}|,\end{aligned}\quad (22)$$

Also, (14) leads to

$$\Lambda(|u_\theta\rangle\langle u_{\theta'}|) = \sum_i \alpha_{\theta,i} \overline{\alpha_{\theta',i}} |v_\theta\rangle\langle v_{\theta'}| = H_{\theta,\theta'} |v_\theta\rangle\langle v_{\theta'}|. \quad (23)$$

Inserting (23) to (22) and equating the coefficients, we have

$$\langle u_\theta | u_{\theta'} \rangle H_{\theta,\theta'} = \langle v_\theta | v_{\theta'} \rangle. \quad (24)$$

Suppose

$$\langle u_\theta | u_{\theta'} \rangle \neq 0.$$

Then, by Lemma 21,

$$|u_\theta\rangle\langle u_{\theta'}| = \frac{1}{\langle u_\theta | u_{\theta'} \rangle} |u_\theta\rangle\langle u_\theta| \cdot |u_{\theta'}\rangle\langle u_{\theta'}| \in \mathcal{M}_\Lambda.$$

Therefore,

$$\begin{aligned}\langle u_\theta | u_{\theta'} \rangle |v_\theta\rangle\langle v_{\theta'}| &= \langle u_\theta | u_{\theta'} \rangle \Lambda^* (|u_\theta\rangle\langle u_\theta|) \\ &= \Lambda(|u_\theta\rangle\langle u_\theta| \cdot |u_{\theta'}\rangle\langle u_{\theta'}|) = \Lambda(|u_\theta\rangle\langle u_\theta|) \Lambda^* (|u_{\theta'}\rangle\langle u_{\theta'}|) \\ &= |v_\theta\rangle\langle v_\theta| \Lambda(|u_{\theta'}\rangle\langle u_{\theta'}|).\end{aligned}$$

Inserting (23) into the above equation and equating the coefficients, we obtain

$$\langle u_\theta | u_{\theta'} \rangle = \langle v_\theta | v_{\theta'} \rangle H_{\theta',\theta}. \quad (25)$$

On the other hand, if $\langle u_\theta | u_{\theta'} \rangle = 0$, $\langle v_\theta | v_{\theta'} \rangle = 0$ by (24). Thus, (25) holds in this case, too.

By Theorem 1, (24) and (25) means that \mathcal{U} and \mathcal{V} are convertible by CP trace preserving maps back and forth. Therefore, by Theorem 1, we have the assertion. ■

Using theory of operator algebra more intensively, we give another proof of Theorem 11 below. Denote by $\left[\left[\hat{\mathcal{E}}\right]\right]$ and $\left[\left[\hat{\mathcal{F}}\right]\right]$ the *-algebra generated by $\hat{\mathcal{E}} = \{L_\theta\}_{\theta \in \Theta}$ and $\hat{\mathcal{F}} = \{M_\theta\}_{\theta \in \Theta}$, respectively. Since each of them is a finite dimensional representation of a finite dimensional C^* -algebra, by Lemma 23, for some unitary operators U_1 and U_2 ,

$$\begin{aligned}\left[\left[\hat{\mathcal{E}}\right]\right] &= U_1 \bigoplus_{n \in \mathbb{N}} \mathcal{L}(\mathbb{C}^n) \otimes I_{d_{1,n}} U_1^\dagger, \\ \left[\left[\hat{\mathcal{F}}\right]\right] &= U_2 \bigoplus_{n \in \mathbb{N}} \mathcal{L}(\mathbb{C}^n) \otimes I_{d_{2,n}} U_2^\dagger.\end{aligned}$$

Here, we used the convention that $d_{1,n} = 0$ in $\left[\left[\hat{\mathcal{E}}\right]\right]$ does not have a component isomorphic to $\mathcal{L}(\mathbb{C}^n)$. $P_{1,n}$ and $P_{2,n}$ denotes the projection onto the subspace $U_1(\mathbb{C}^n \otimes I_{d_{1,n}})$ and $U_2(\mathbb{C}^n \otimes I_{d_{2,n}})$, respectively.

Proof. Since each L_θ is a rank-1 operator, $d_{1,n}$ is 0 or 1, for all n . So

$$\left[\left[\hat{\mathcal{E}}\right]\right] = U_1 \bigoplus_{n \in \mathbb{N}, d_{1,n} \neq 0} \mathcal{L}(\mathbb{C}^n) U_1^\dagger,$$

By (3) and Lemmas 10, Λ^* is *-homomorphism from $\left[\left[\hat{\mathcal{E}}\right]\right]$ onto $\left[\left[\hat{\mathcal{F}}\right]\right]$, or is a representation of $\left[\left[\hat{\mathcal{E}}\right]\right]$. Hence, by Lemma 23,

By Lemma 10 and (3), Λ^* is *-homomorphism from $\left[\left[\hat{\mathcal{E}}\right]\right]$ onto $\left[\left[\hat{\mathcal{F}}\right]\right]$, and thus, it is a representation of $\left[\left[\hat{\mathcal{E}}\right]\right]$ on the finite dimensional Hilbert space \mathcal{K} . Therefore, by Lemma 23, the restriction of Λ^* to $\left[\left[\hat{\mathcal{E}}\right]\right]$ is unitary equivalent to

$$\bigoplus_{n \in \mathbb{N}, d_{1,n} \neq 0} \mathbf{I}_{\mathbb{C}^n}^{(d_{2,n})}.$$

In fact, as is shown below, $d_{2,n} \neq 0$ if $d_{1,n} \neq 0$.

Suppose there is an n_0 such that $d_{1,n_0} \neq 0$ and $d_{2,n} = 0$. Observe that $\text{rank } L_\theta = 1$ for all θ implies that $P_{1,n} L_\theta P_{1,n} \neq 0$ holds only for a single n for each θ . Therefore, there is at least one θ such that $L_\theta \in U_1 \mathcal{L}(\mathbb{C}^{n_0}) U_1^\dagger$. Since $n_0 \notin N_{\Lambda^*}$ means $\Lambda^*(L_\theta) = M_\theta = 0$, we have contradiction. Therefore, $d_{2,n} \neq 0$ if $d_{1,n} \neq 0$.

Finally, observe that $d_{2,n}$ is 0 or 1 by the same reason as $d_{1,n}$ is 0 or 1. This means that restriction of Λ^* to $\hat{\mathcal{E}}$ is unitary equivalent to the identity operator. Therefore, $\hat{\mathcal{E}}$ and $\hat{\mathcal{F}}$ are unitary equivalent. ■

4 Projectors

Modifying the second proof of the above theorem slightly, we obtain similar result for the case where $\text{rank } L_\theta = 1$ and M_θ is a constant multiple of a projector.

Let us divide $\hat{\mathcal{E}}$ into $\bigcup_{\kappa} \hat{\mathcal{E}}_{\kappa}$, so that the following conditions are satisfied; there is a sequence $\theta = \theta_1, \theta_2, \dots, \theta_k = \theta'$ with $L_{\theta_i} L_{\theta_{i+1}} \neq 0$, $i = 1, \dots, k$ for any $L_\theta, L_{\theta'} \in \hat{\mathcal{E}}_{\kappa}$, while $L_\theta L_{\theta'} = 0$ for any $L_\theta \in \hat{\mathcal{E}}_{\kappa}$ and $L_{\theta'} \in \hat{\mathcal{E}}_{\kappa'}$. Then,

$$\left[\left[\hat{\mathcal{E}} \right] \right] = \bigoplus_{\kappa} \left[\left[\hat{\mathcal{E}}_{\kappa} \right] \right].$$

It is easy to check that $\left[\left[\hat{\mathcal{E}}_{\kappa} \right] \right]$ does not break into direct sum of smaller subalgebras. In fact, by the first condition, if $|u_\theta\rangle\langle u_\theta|, |u_{\theta'}\rangle\langle u_{\theta'}|$ are the elements of $\left[\left[\hat{\mathcal{E}}_{\kappa} \right] \right]$, so are $|u_\theta\rangle\langle u_{\theta'}|$ and $|u_{\theta'}\rangle\langle u_\theta|$. Thus, $\left[\left[\hat{\mathcal{E}}_{\kappa} \right] \right]$ is nothing but the linear operators on the space spanned by $\mathcal{U} = \{u_\theta\}_{\theta \in \Theta}$.

Proposition 12 *Suppose that $L_\theta = |u_\theta\rangle\langle u_\theta|$, M_θ is a constant multiple of projector, and $\|L_\theta\| = \|M_\theta\|$ for each θ . Then, there is a CP subunital map Λ satisfying (3) exists if and only if there is a system $\mathcal{V} = \{v_\theta\}_{\theta \in \Theta}$ of vectors such that \mathcal{V} is unitary equivalent to $\mathcal{U} = \{u_\theta\}_{\theta \in \Theta}$ and that $M_\theta = |v_\theta\rangle\langle v_\theta| \otimes I_{d_\kappa}$ ($L_\theta \in \hat{\mathcal{E}}_{\kappa}$).*

When L_θ is also not of rank 1, still one can state something. The proof of the following proposition is straightforward, thus omitted.

Proposition 13 *Suppose that L_θ, M_θ is a constant multiple of projector, and $\|L_\theta\| = \|M_\theta\|$ for each θ . Then, there is a CP subunital map Λ satisfying (3) exists if and only if*

$$\begin{aligned} L_\theta &= U_1 \bigoplus_{n \in \mathbb{N}} a_{n,\theta} I_n \otimes I_{d_{1,n}} U_1^\dagger, \\ M_\theta &= U_2 \bigoplus_{n \in \mathbb{N}} b_{n,\theta} I_n \otimes I_{d_{2,n}} U_2^\dagger, \end{aligned}$$

where U_1, U_2 are unitary, and $b_{n,\theta} \neq 0$ and $d_{2,n} \neq 0$ unless $a_{n,\theta} \neq 0$ or $d_{1,n} = 0$.

5 From rank-1 operators to arbitrary operators

Theorem 14 *Suppose $\mathcal{U} = \{u_\theta\}_{\theta \in \Theta}$ is linearly independent. Then there is a CP subunital map Λ with (3) exists if and only if there are operators $M_{\theta,\theta'} \in \mathcal{L}(K)$ ($\theta, \theta' \in \Theta$) such that*

$$\sum_{\theta, \theta' \in \Theta} M_{\theta,\theta'} \otimes |e_\theta\rangle\langle e_{\theta'}| \geq 0, \quad M_{\theta,\theta} = M_\theta \quad (26)$$

$$\sum_{\theta, \theta' \in \Theta} M_{\theta,\theta'} (G_{\mathcal{U}}^{-1})_{\theta,\theta'} \leq I_K, \quad (27)$$

where $\{e_\theta\}_{\theta \in \Theta}$ is an orthonormal system of vectors spanning a Hilbert space \mathcal{K}' .

Also, there is a CP unital map Λ with (3) exists if and only if there are operators $M_{\theta, \theta'} \in \mathcal{L}(\mathcal{K})$ ($\theta, \theta' \in \Theta$) with (26) and

$$\sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} (G_U^{-1})_{\theta, \theta'} = I_{\mathcal{K}}. \quad (28)$$

Proof. Λ is CP if and only if

$$\sum_{\theta, \theta' \in \Theta} \Lambda(|u_\theta\rangle\langle u_{\theta'}|) \otimes |e_\theta\rangle\langle e_{\theta'}| = \sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} \otimes |e_\theta\rangle\langle e_{\theta'}| \geq 0,$$

where we put

$$M_{\theta, \theta'} = \Lambda(|u_\theta\rangle\langle u_{\theta'}|).$$

Observe

$$\begin{aligned} \sum_{\theta, \theta' \in \Theta} \Lambda(|u_\theta\rangle\langle u_{\theta'}|) \otimes |e_\theta\rangle\langle e_{\theta'}| &= \sum_{\theta, \theta' \in \Theta} \Lambda^*([\mathcal{U}]|e_\theta\rangle\langle e_{\theta'}|[\mathcal{U}]^\dagger) \otimes |e_\theta\rangle\langle e_{\theta'}| \\ &= \sum_{\theta, \theta' \in \Theta} \Lambda^* (|e_\theta\rangle\langle e_{\theta'}|) \otimes [\mathcal{U}]^T |e_\theta\rangle\langle e_{\theta'}| [\overline{\mathcal{U}}]. \end{aligned}$$

Equating this with $\sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} \otimes |e_\theta\rangle\langle e_{\theta'}|$ and solving about $\sum_{\theta, \theta' \in \Theta} \Lambda(|e_\theta\rangle\langle e_{\theta'}|) \otimes |e_\theta\rangle\langle e_{\theta'}|$, we have

$$\sum_{\theta, \theta' \in \Theta} \Lambda(|e_\theta\rangle\langle e_{\theta'}|) \otimes |e_\theta\rangle\langle e_{\theta'}| = \sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} \otimes ([\mathcal{U}]^T)^{-1} |e_\theta\rangle\langle e_{\theta'}| ([\overline{\mathcal{U}}])^{-1}.$$

Therefore, Λ is subunital if and only if

$$\begin{aligned} I_{\mathcal{K}} &\geq \text{tr}_{\mathcal{K}'} \sum_{\theta, \theta' \in \Theta} \Lambda(|e_\theta\rangle\langle e_{\theta'}|) \otimes |e_\theta\rangle\langle e_{\theta'}| \\ &= \sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} \otimes ([\overline{\mathcal{U}}])^{-1} ([\mathcal{U}]^T)^{-1} |e_\theta\rangle\langle e_{\theta'}| \\ &= \sum_{\theta, \theta' \in \Theta} M_{\theta, \theta'} (G_U^{-1})_{\theta, \theta'}. \end{aligned}$$

The map Λ is unital if and only if the equality in the above inequality holds. Thus we have (28). ■

Remark 15 Suppose $M_\theta = |v_\theta\rangle\langle v_\theta|$, for each $\theta \in \Theta$. Then, by (23), (26) and (27) become

$$\begin{aligned} \sum_{\theta, \theta' \in \Theta} H_{\theta, \theta'} |v_\theta\rangle\langle v_{\theta'}| \otimes |e_\theta\rangle\langle e_{\theta'}| &\geq 0, \\ \sum_{\theta, \theta' \in \Theta} |v_\theta\rangle\langle v_{\theta'}| H_{\theta, \theta'} (G_U^{-1})_{\theta, \theta'} &\leq I_{\mathcal{K}}, \end{aligned}$$

respectively. The first inequality is verified by

$$\sum_{\theta, \theta' \in \Theta} H_{\theta, \theta'} |v_\theta\rangle \langle v_{\theta'}| \otimes |e_\theta\rangle \langle e_{\theta'}| = A \sum_{\theta, \theta' \in \Theta} |v_1\rangle \langle v_1| \otimes H_{\theta, \theta'} |e_\theta\rangle \langle e_{\theta'}| A^\dagger \geq 0,$$

where

$$A := \sum_{\theta \in \Theta} |v_\theta\rangle \langle v_1| \otimes |e_\theta\rangle \langle e_\theta|.$$

The second inequality can be rewritten as

$$\begin{aligned} I_{\mathcal{K}} &\geq \sum_{\theta, \theta' \in \Theta} |v_\theta\rangle \langle v_{\theta'}| H_{\theta, \theta'} (G_{\mathcal{U}}^{-1})_{\theta, \theta'} \\ &= [\mathcal{V}] H \circ (G_{\mathcal{U}}^{-1}) [\mathcal{V}]^\dagger. \end{aligned}$$

Hence, if $\mathcal{V} = \{v_\theta\}_{\theta \in \Theta}$ is linearly independent, we obtain (15).

Inserting some $M_{\theta, \theta'}$ satisfying (26) into (27), one obtain sufficient condition for (3) to hold for a CP subunital map Λ^* . For example,

$$\left\| \sum_{\theta \in \Theta} M_\theta (G_{\mathcal{U}}^{-1})_{\theta, \theta} \right\| \leq 1,$$

or

$$\left\| \sum_{\theta, \theta' \in \Theta} \sqrt{M_\theta} \sqrt{M_{\theta'}} (G_{\mathcal{U}}^{-1})_{\theta, \theta'} \right\| \leq 1,$$

and so on.

But obtaining the necessary and sufficient condition for (26) and (27), or for (26) and (28), is quite non-trivial task, in general. So in the next subsection, we deal with an easy case, the case of $|\Theta| = 2$ and $\dim \mathcal{H} = \dim \mathcal{K} = 2$.

6 2-dimensional and $|\Theta| = 2$ case

In this section, we work on the case of $|\Theta| = 2$ and $\dim \mathcal{H} = \dim \mathcal{K} = 2$. First, we note that the problem is reduce to the case of $\text{rank } L_\theta = 1$ and $\|L_\theta\| = 1$ ($\theta = 1, 2$). Observe that (3) for a CP map Λ is equivalent to

$$\begin{aligned} \Lambda(L_1 - t_1 L_2) &= M_1 - t_1 M_2, \\ \Lambda(L_2 - t_2 L_1) &= M_2 - t_2 M_1. \end{aligned}$$

Choose t_1 and t_2 so that $L_1 - t_1 L_2$ and $L_2 - t_2 L_1$ is rank-1 positive operator, the problem is reduced to the case of $\text{rank } L_\theta = 1$ ($\theta = 1, 2$). Since multiplying constant to the input and the output does not change the problem, $\|L_\theta\| = 1$ ($\theta = 1, 2$) can be assumed without loss of generality, too.

Remark 16 In case that Λ is unital, there is another way of reducing the problem to the case of $\text{rank } L_\theta = 1$ and $\|L_\theta\| = 1$ ($\theta = 1, 2$). Observe that (3) for a CP unital map Λ is the same as

$$\Lambda(a_\theta(L_\theta - b_\theta I_{\mathcal{H}})) = a_\theta(M_\theta - b_\theta I_{\mathcal{K}}),$$

due to the fact that Λ is linear and unital. Therefore, choosing $b_\theta := \|L_\theta\|$, we can reduce the problem to the case where $L_\theta = |u_\theta\rangle\langle u_\theta|$ for each $\theta \in \Theta$. Note here that by (44) $M_\theta - \|L_\theta\| I_{\mathcal{K}}$ is positive definite. Also, by choosing a_θ properly, we can assume, without loss of generality, $\|u_1\| = \|u_2\| = 1$.

Also, $\{u_1, u_2\}$ is assumed to be linearly independent, since otherwise the problem becomes trivial. We choose the phase of u_θ so that $\langle u_1 | u_2 \rangle = c \geq 0$. Also, we suppose $M_1 > 0$. (when both of M_1 and M_2 are not strictly positive, they are rank-1, and thus reduce to the case of $\text{rank } M_\theta = 1$.)

Below, we study the condition that (3) holds for a CP unital map Λ , or rewrite (26) and (28) in more ‘tractable’ expression.

(28) is the same as

$$M_1 + M_2 - c(M_{1,2} + M_{2,1}) = (1 - c^2) I_{\mathcal{K}},$$

or equivalently,

$$M_1^{-1/2} M_{2,1} M_1^{-1/2} = M_0 + \sqrt{-1} B,$$

where

$$M_0 := \frac{1}{2c} M_1^{-1/2} (M_1 + M_2 - (1 - c^2) I_{\mathcal{K}}) M_1^{-1/2}$$

and B is a Hermitian operator. Therefore, (26) can be rewritten as

$$\begin{aligned} (M_0 + \sqrt{-1} B) (M_0 - \sqrt{-1} B) &= M_1^{-1/2} M_{2,1} M_1^{-1} M_{1,2} M_1^{-1/2} \\ &\leq M_1^{-1/2} M_2 M_1^{-1/2}. \end{aligned}$$

Defining

$$\mathcal{C} := \{M; M \geq B^2 + \sqrt{-1} [B, M_0]\},$$

the above inequality is equivalent to

$$M_1^{-1/2} M_2 M_1^{-1/2} - M_0^2 \in \mathcal{C}.$$

So our task is to find a convenient expression of the set \mathcal{C} .

In the previous paper, we have already have solved this problem (Appendix C of [8]). Choose an orthonormal basis of \mathcal{K} so that M_0 defined above is diagonalized. Let us parameterize M_0 and $M_1^{-1/2} M_2 M_1^{-1/2} - M_0^2$ as follows,

$$\begin{aligned} M_0 &= l\sigma_z + mI_2, \\ M_1^{-1/2} M_2 M_1^{-1/2} - M_0^2 &= l^2 (x\sigma_x + y\sigma_y + z\sigma_z + wI_{\mathcal{K}}), \end{aligned}$$

where σ_x , σ_y , and σ_z are Pauli matrices. Then,

$$\begin{aligned}
& M_1^{-1/2} M_2 M_1^{-1/2} - M_0^2 \in \mathcal{C} \\
& \Leftrightarrow \exists s \in [-2, 2], \quad x' \sigma_x + z \sigma_z + w I_{\mathcal{K}} \geq s \sigma_x + \frac{s^2}{4} I_{\mathcal{K}} \\
& \Leftrightarrow z = 0, \quad w \geq f_1(x') \\
& \text{or } z \neq 0, w \geq f_2(x', z), \quad f_3(x', z, w) \geq 0
\end{aligned} \tag{29}$$

where we have defined

$$\begin{aligned}
x' &:= \sqrt{x^2 + y^2} \\
f_1(x) &:= \begin{cases} |x| - 1, & (|x| \geq 2) \\ \frac{1}{4}x^2, & (|x| \leq 2) \end{cases}, \\
f_2(x, z) &:= \begin{cases} \sqrt{x^2 + z^2} - 1, & (x^2 + z^2 \geq 4) \\ \frac{1}{4}\{x^2 + z^2\}, & (x^2 + z^2 \leq 4) \end{cases},
\end{aligned}$$

and

$$\begin{aligned}
& f_3(x, z, w) \\
&:= 16w^4 + (-8x^2 + 8z^2 + 32)w^3 + (x^4 + 2x^2z^2 - 32x^2 + z^4 - 8z^2 + 16)w^2 \\
& \quad (30) \\
& + (10x^4 + 2x^2z^2 - 8x^2 - 8z^4 - 32z^2)w + (x^4 - 3x^4z^2 - x^6 - 3x^2z^4 + 20x^2z^2 - z^6 - 8z^4 - 16z^2).
\end{aligned}$$

The quantities l , x' , z , and w , which are needed to check (29), can be computed directly by the following formulas:

$$\begin{aligned}
l &= \frac{1}{2} \sqrt{(\text{tr } M_0)^2 - 4(\det M_0)^2}, \\
w &= \frac{1}{2l^2} \text{tr } M, \\
z &= \frac{1}{2l^3} \text{tr} \left\{ M \left(M_0 - \frac{1}{2} (\text{tr } M_0) I_{\mathcal{K}} \right) \right\} = \frac{1}{2l^3} \left(\text{tr } M M_0 - \frac{1}{2} (\text{tr } M_0) (\text{tr } M) \right), \\
x' &= \sqrt{\text{tr} \left\{ \frac{1}{l^2} M - \frac{z}{l} M_0 + \left(\frac{z}{2l} (\text{tr } M_0) - w \right) I_{\mathcal{K}} \right\}^2},
\end{aligned}$$

where $M := M_1^{-1/2} M_2 M_1^{-1/2} - M_0^2$.

7 Randomization criteria

In this section, Θ is any set.

Lemma 17 (*Fan's minimax theorem, [2]*) Suppose that \mathcal{X} be a compact convex subset of vector space, and \mathcal{Y} be a convex subset of a vector space. Assume that $f : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ satisfies following conditions: (1) $x \rightarrow f(x, y)$ is lower semi

continuous and convex on \mathcal{X} for every $y \in \mathcal{Y}$: (2) $y \rightarrow f(x, y)$ is concave on \mathcal{Y} for every $x \in \mathcal{X}$. Then

$$\min_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y) = \sup_{y \in \mathcal{Y}} \min_{x \in \mathcal{X}} f(x, y).$$

Theorem 18 (randomization criteria) Let $e_\theta \geq 0$, $\theta \in \Theta$. There is a CP (sub)unital map Λ satisfying

$$\forall \theta \in \Theta, \quad \|\Lambda(L_\theta) - M_\theta\| \leq e_\theta \quad (31)$$

(3) if and only if

$$\inf_{\Lambda_1} \sum_{\theta \in \Theta_0} p_\theta \text{tr} \Lambda_1(L_\theta) X_\theta \leq \inf_{\Lambda_2} \sum_{\theta \in \Theta_0} p_\theta \text{tr} \Lambda_2(M_\theta) X_\theta + e_\theta \quad (32)$$

holds for any subset Θ_0 of Θ with $|\Theta_0| < \infty$, any probability distribution $\{p_\theta\}_{\theta \in \Theta_0}$ on Θ_0 , and any family of operators $\{X_\theta\}_{\theta \in \Theta}$ on \mathcal{K} with $\|X_\theta\|_1 \leq 1$, $\forall \theta \in \Theta$. Here, Λ_1 and Λ_2 moves over the set of all CP (sub)unital maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$ and $\mathcal{L}(\mathcal{K})$ to $\mathcal{L}(\mathcal{K})$, respectively. This in turn equivalent to that

$$\sup_{\Lambda_1} \sum_{\theta \in \Theta_0} p_\theta \text{tr} \Lambda_1(L_\theta) X_\theta + e_\theta \geq \sup_{\Lambda_2} \sum_{\theta \in \Theta_0} p_\theta \text{tr} \Lambda_2(M_\theta) X_\theta \quad (33)$$

holds for any subset Θ_0 of Θ with $|\Theta_0| < \infty$, any probability distribution $\{p_\theta\}_{\theta \in \Theta_0}$ on Θ_0 , and any family of operators $\{X_\theta\}_{\theta \in \Theta}$ on \mathcal{K} with $\|X_\theta\|_1 \leq 1$, $\forall \theta \in \Theta$.

Proof. Since 'only if' part of the statement is trivial, we prove 'if' part. Let

$$f_1(\Lambda_1, \{X_\theta\}) := \int \text{tr} \Lambda_1(L_\theta) X_\theta \, dp(\theta) - \int \text{tr} M_\theta X_\theta \, dp(\theta),$$

where p is a measure whose support is with finite cardinality (The support of p is Θ_0). Obviously, f_1 is bilinear and continuous. The set of all CP (sub)unital maps is obviously compact, since \mathcal{H} and \mathcal{K} are finite dimensional (even if they are infinite dimensional separable Hilbert space, the set is compact with respect to a topology which makes f_1 continuous in Λ_1 .)

Also, by the assumption, $\min_{\Lambda_1} f_1(\Lambda_1, \{X_\theta\}) \leq 0$ for each $\{X_\theta\}_{\theta \in \Theta}$ (let Λ_2 be the identity map in (32)). Therefore, by Lemma 17,

$$\begin{aligned} e_\theta &\geq \sup \left\{ \min_{\Lambda_1} f_1(\Lambda_1, \{X_\theta\}); \|X_\theta\|_1 \leq 1 \right\} \\ &= \min_{\Lambda_1} \sup \{ f_1(\Lambda_1, \{X_\theta\}); \|X_\theta\|_1 \leq 1 \} \\ &= \min_{\Lambda_1} \sup \left\{ \sum_{\theta \in \text{supp } p} \text{tr} (\Lambda_1(L_\theta) - M_\theta) X_\theta; \|X_\theta\|_1 \leq 1 \right\} \\ &= \min_{\Lambda_1} \int \|\Lambda_1(L_\theta) - M_\theta\| \, dp(\theta). \end{aligned}$$

Next, let

$$f_2(\Lambda_1, p) := \int \|\Lambda_1(L_\theta) - M_\theta\| dp(\theta),$$

which is lower semi-continuous, convex in Λ_1 , and linear in p . By Lemma 17,

$$\begin{aligned} e_\theta &\geq \sup_{p: |\text{supp } p| < \infty} \min_{\Lambda_1} \int \|\Lambda_1(L_\theta) - M_\theta\| dp(\theta) \\ &= \min_{\Lambda_1} \sup_{p: |\text{supp } p| < \infty} \int \|\Lambda_1(L_\theta) - M_\theta\| dp(\theta) \\ &= \min_{\Lambda_1} \sup_{\theta \in \Theta} \|\Lambda_1(L_\theta) - M_\theta\|, \end{aligned}$$

which means existence of (sub)unital map satisfying (31). ■

8 Commutative case

In this section we suppose that

$$L_\theta = \sum_{i=1}^{d_1} l_{\theta,i} |e_i\rangle \langle e_i|, \quad M_\theta = \sum_{i=1}^{d_2} m_{\theta,i} |f_i\rangle \langle f_i|, \quad (34)$$

where $\{e_i\}$ and $\{f_i\}$ are a complete orthonormal basis of \mathcal{H} and \mathcal{K} , respectively.

Theorem 19 *Suppose (34) holds. Then there is a CP unital map Λ satisfying (3) if and only if for any set $\{x_\theta\}_{\theta \in \Theta}$ of real numbers*

$$\forall \{x_\theta\}_{\theta \in \Theta}, x_\theta \in \mathbb{R}, \lambda_{\max} \left(\sum_{\theta \in \Theta} x_\theta L_\theta \right) \geq \lambda_{\max} \left(\sum_{\theta \in \Theta} x_\theta M_\theta \right). \quad (35)$$

This is equivalent to

$$\forall \{x_\theta\}_{\theta \in \Theta}, x_\theta \in \mathbb{R}, \lambda_{\min} \left(\sum_{\theta \in \Theta} x_\theta L_\theta \right) \leq \lambda_{\min} \left(\sum_{\theta \in \Theta} x_\theta M_\theta \right). \quad (36)$$

Also, there is a CP subunital map satisfying (3) if and only if

$$\forall \{x_\theta\}_{\theta \in \Theta}, x_\theta \in \mathbb{R}, \max \left\{ \lambda_{\max} \left(\sum_{\theta \in \Theta} x_\theta L_\theta \right), 0 \right\} \geq \max \left\{ \lambda_{\max} \left(\sum_{\theta \in \Theta} x_\theta M_\theta \right), 0 \right\}, \quad (37)$$

or equivalently,

$$\forall \{x_\theta\}_{\theta \in \Theta}, x_\theta \in \mathbb{R}, \min \left\{ \lambda_{\min} \left(\sum_{\theta \in \Theta} x_\theta L_\theta \right), 0 \right\} \geq \min \left\{ \lambda_{\min} \left(\sum_{\theta \in \Theta} x_\theta M_\theta \right), 0 \right\}, \quad (38)$$

Proof. By Theorem 18, existence of CP unital map Λ satisfying (3) is equivalent to

$$\sup_P \sum_{i,j,\theta} x_{\theta,j} P_{j,i} l_{\theta,i} \geq \sup_P \sum_{i,j,\theta} x_{\theta,j} P_{j,i} m_{\theta,i}, \quad (39)$$

where P moves over the set of all column stochastic matrices, $P_{j,i} \geq 0$, $\sum_i P_{j,i} = 1$. Observe

$$\sup_P \sum_{i,j,\theta} x_{\theta,j} P_{j,i} l_{\theta,i} = \sum_j \max_i \sum_{\theta} x_{\theta,j} l_{\theta,i}.$$

Hence,

$$\sum_j \lambda_{\max} \left(\sum_{\theta \in \Theta} x_{\theta,j} L_{\theta} \right) \geq \sum_j \lambda_{\max} \left(\sum_{\theta \in \Theta} x_{\theta,j} M_{\theta} \right)$$

holds for any $x_{\theta,j}$. This is equivalent to (35). (36) is obtained by replacing x_{θ} by $-x_{\theta}$.

When Λ is subunital, P in (39) is column sub stochastic, $P_{j,i} \geq 0$, $\sum_i P_{j,i} \leq 1$. Therefore,

$$\sup_P \sum_{i,j,\theta} x_{\theta,j} P_{j,i} l_{\theta,i} = \sum_j \max_i \left\{ 0, \sum_{\theta} x_{\theta,j} l_{\theta,i} \right\}.$$

Thus we obtain (37). (36) is obtained by replacing x_{θ} by $-x_{\theta}$. ■

Note that the condition

$$\forall \{x_{\theta}\}_{\theta \in \Theta}, x_{\theta} \in \mathbb{R}, \left\| \sum_{\theta \in \Theta} x_{\theta} L_{\theta} \right\| \geq \left\| \sum_{\theta \in \Theta} x_{\theta} M_{\theta} \right\|. \quad (40)$$

is a necessary condition of (35). In fact, by

$$\|L\| = \max \{ \lambda_{\max}(L), -\lambda_{\min}(L) \}, \quad (41)$$

combining (35) and (36) leads to (40). Thus, (35) implies (40). But, suppose $|\Theta| = 1$ and $\lambda_{\max}(L_1) \leq -\lambda_{\min}(L_1)$. Then $\|L_1\| = -\lambda_{\min}(L_1)$, and

$$\begin{aligned} \|-L_1\| &= \max \{ \lambda_{\max}(-L_1), -\lambda_{\min}(-L_1) \} \\ &= \max \{ -\lambda_{\min}(L_1), \lambda_{\max}(L_1) \} \\ &= -\lambda_{\min}(L_1), \end{aligned}$$

both of which are not related to $\lambda_{\max}(L_1)$. Thus if $\lambda_{\max}(L_1) \leq -\lambda_{\min}(L_1)$ and $\lambda_{\max}(M_1) \leq -\lambda_{\min}(M_1)$, (40) cannot be a sufficient condition.

Another necessary condition is

$$\forall \{x_{\theta}\}_{\theta \in \Theta}, x_{\theta} \in \mathbb{R}, \operatorname{sp} \left(\sum_{\theta \in \Theta} x_{\theta} L_{\theta} \right) \leq \operatorname{sp} \left(\sum_{\theta \in \Theta} x_{\theta} M_{\theta} \right). \quad (42)$$

Observe that one of $\lambda_{\max}(L) = \|L\|$ or $\lambda_{\max}(L) = \operatorname{sp}(L) - \|L\|$ is always true. Therefore, the combination of (40) and (42) is equivalent to (35).

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A Monotone functionals

Let C be an arbitrary Hermitian operator on \mathcal{H} . Suppose

$$\lambda_{\max}(C) \geq 0 \geq \lambda_{\min}(C). \quad (43)$$

Observe

$$\mathrm{tr} \Lambda^*(\rho) = \mathrm{tr} \rho \Lambda(I_{\mathcal{H}}) \leq \mathrm{tr}(\rho \cdot I_{\mathcal{K}}) = 1,$$

holds for any CP subunital Λ . Therefore, if $\lambda_{\max}(C) \geq 0$, we have

$$\begin{aligned}
\lambda_{\max}(\Lambda(C)) &= \max_{\rho: \rho \geq 0, \text{tr } \rho = 1} \text{tr } \rho \Lambda(C) \\
&\leq \max_{\rho: \rho \geq 0, \text{tr } \rho \leq 1} \text{tr } \rho \Lambda(C) \\
&= \max_{\rho: \rho \geq 0, \text{tr } \rho \leq 1} \text{tr } \Lambda^*(\rho) C \\
&\leq \max_{\rho: \rho \geq 0, \text{tr } \rho \leq 1} \text{tr } \rho C \\
&= \max_{\rho: \rho \geq 0, \text{tr } \rho = 1} \text{tr } \rho C \\
&= \lambda_{\max}(C).
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{\min}(\Lambda(C)) &= \min_{\rho: \rho \geq 0, \text{tr } \rho = 1} \text{tr } \rho \Lambda(C) \\
&= \min_{\rho: \rho \geq 0, \text{tr } \rho = 1} \text{tr } \Lambda^*(\rho) C \\
&\geq \min_{\rho: \rho \geq 0, \text{tr } \rho = 1} \text{tr } \rho C \\
&\geq \lambda_{\min}(C).
\end{aligned}$$

If Λ is CP and unital, these inequality holds without the restriction (43), due to almost parallel argument.

Since $\|C\| = \max\{\lambda_{\max}(C), -\lambda_{\min}(C)\}$ for a Hermitian operator C , the inequality

$$\|C\| \geq \|\Lambda(C)\| \quad (44)$$

holds for any CP subunital map Λ and any Hermitian operator C without the restriction (43).

B Multiplicative domain and finite dimensional C^* -algebra

This section is based on Section 3 of [7]. When Λ is completely positive from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$, we have Schwartz inequality

$$\Lambda(L^\dagger) \Lambda(L) \leq \|\Lambda\| \Lambda(L^\dagger L). \quad (45)$$

The multiplicative domain \mathcal{M}_Λ of completely positive map Λ is a set of operators on \mathcal{H} such that

$$\mathcal{M}_\Lambda := \{\Lambda(L^\dagger) \Lambda(L) = \|\Lambda\| \Lambda(L^\dagger L)\}.$$

Remark 20 When Λ is a positive map which may not be 2-positive, we have to replace $\|\Lambda^*\|$ in the above expressions with $\|\Lambda^*\|_S$ defined by

$$\|\Lambda\|_S := \inf \{c; \Lambda(L^\dagger) \Lambda(L) \leq c \Lambda(L^\dagger L), L \in \mathcal{L}(\mathcal{H})\}$$

But as is remarked in Section 3 of [7], $\|\Lambda\| = \|\Lambda\|_S$ when Λ is 2-positive.

Lemma 21 (Lemma 3.9 of [7]) Let Λ be a completely positive map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. \mathcal{M}_Λ is a vector space closed by multiplication and \dagger , or constitutes a $*$ -algebra. Also, if $L \in \mathcal{M}_\Lambda^*$, for any $M \in \mathcal{L}(\mathcal{H})$

$$\Lambda(L)\Lambda(M) = \|\Lambda\| \Lambda(LM).$$

Lemma 22 (Corollary 2.3.8 of [1]) Let Λ be a positive map from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{K})$. Then, $\|\Lambda\| = \|\Lambda(I_\mathcal{H})\|$.

Lemma 23 (Theorem III.1.1 and Corollary III.2.1 of [6]) Any finite dimensional C^* -algebra is $*$ -isomorphic to

$$\mathcal{L}(\mathbb{C}^{n_1}) \oplus \cdots \oplus \mathcal{L}(\mathbb{C}^{n_k}).$$

Also, If π is a non-degenerate $*$ -representation of a finite dimensional C^* -algebra above, then there are cardinal numbers d_1, \dots, d_k so that it is unitarily equivalent to $\mathbf{I}_{\mathbb{C}^{n_1}}^{(d_1)} \oplus \cdots \oplus \mathbf{I}_{\mathbb{C}^{n_k}}^{(d_k)}$.